Exercise 1:

Something to prove:
Show that the arithmetic mean value of two numbers a and b which are both larger than zero is always larger or equal to the geometric mean value of these numbers:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Tip: Remove the square root, shift everything to left hand side … >= 0, then use the formulas for square of sums and differences to get some quadratic expression, which is obviously always > zero.

Exercise 2:

Geometric series:
The factor denoting the ratio between subsequent fret distances of the guitar is

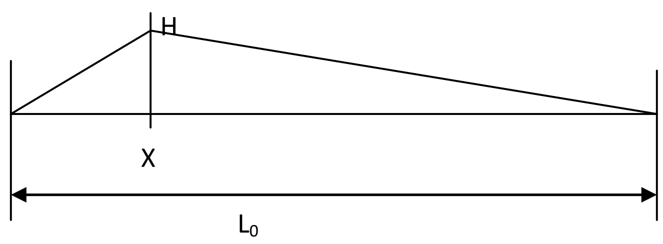
$$\frac{1}{\sqrt[12]{2}} = 0.943874$$

If the distance from upper bridge to first fret is 5 cm, how long is the fretboard , if 24 frets are desired?

Tip: Use the formula for the sum of the geometric series. It's good, if you know how to derive this formula and also the how to derive the sum of the arithmetic series . It's part of basic mathematical knowledge. Thus have a look on these things, understand and write down in your notes.

Exercise 3:

Differential calculus:
A guitar string of length L_0 is deflected at position $x \cdot L_0$ by a distance H yielding a triangular shape ($0<x<1$).
a) Find an expression for the elongation y of the string. Tip: The elongation, e.g., denoted as y is the ratio between the sum of the "roof" sides of the triangle and the base line of the triangle with length L_0. You can get the length of the roof sides by theorem of Pythagoras. You will get a function y(x). How does it read?
b) At which position we get the smallest value for the elongation y? This is also the place with smallest probability for rupture of the string. Tip: Calculate first derivative of function y with respect to x and set derivative to zero.



Exercise 4:

Quadratic equations.

A cajon has volume

$$V = abc$$

with a and b denoting the short and long side of the front plate, c is depth. Two mechanisms are responsible for sound generation at the cajon: 1) the resonant oscillation of the enclosed air also known as Helmholtz-resonance and 2) the natural frequencies of the vibrating front plate. For the Helmholtz-resonance one gets a relationship, which predicts decreasing frequency with increasing volume

$$f_H = \frac{B}{\sqrt{V}}$$

For the first natural frequency of the oscillating front plate one gets from technical mechanics

$$f_P = A \cdot \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

This frequency decays with side lengths a and b of the front plate. If both frequencies are different, a dissonant overall sound could appear. Thus it is a good idea to choose a and b in such way that f_H and f_P get the same value. Which ratio a/b between length and width of the front plate should be chosen to get same frequencies, if $A = \sqrt{2},\ B = 1$ and $c = \frac{1}{5}$?.

Tip: Introduce volume formula into Helmholtz frequency to get an expression depending on a and b. Then set frequencies equal to one another. Introduce the ratio a/b=x by suitable division and solve for x.

Exercise 5

Minimum and maximum values / differential calculus

You are designing a cajon and you would like to know the highest possible frequency from the oscillation of the front plate for a given area S =ab of the front plate, how should you chose the ratio a/b ?

Tip: Calculate from area S one of the sides a or b and introduce this into the formula for the plate frequency. Then you get an expression for the frequency depending on one side length only. Search the maximum by setting first derivative to zero. Make a sketch of the function for the frequency depending on side length. How can the function be simplified for very small and very large values of the side length?

Exercise 6

Integration:

The neck of an electric guitar can be modeled as rod tightly fixed in the guitar body. A constant bending moment is exerted by the tension of the strings on the neck causing some small curvature of the neck. Clearly this effect should be as small as possible. We want to find the maximum bowing under the string tension load at the upper bridge. Thus we must calculate the bending line of the neck. From technical mechanics it is known that the second derivative of the deflection of a bar is determined by bending moment M , Young's modulus E und the geometrical moment of inertia I_y .

$$w(x)'' = \frac{M}{E \cdot I_y}$$

If total string tension is 400 N and distance from upper edge of the bridge to the center of gravity of neck cross section is 0.015 m we can calculate a bending moment M of 6 Nm. Youngs modulus for maple wood is approximately $E=10^{10}$ N/m^2 and the geometrical moment of inertia for a semi-elliptic cross section of width 0.04 m and height 0.03 m is about $I_y = 10^{-7}$ m^4.

With this information one can calculate the 2nd derivative of the deflection curve as given in the above formula. The bending line is calculated by twofold integration with respect to coordinate x measured along the neck. X=0 should be chosen at the position where the neck is clamped into the body. During integration two integration constants appear. These can be determined from boundary conditions w'(0) =0 und w(0)=0. The first one says that the gradient at the clamping position is zero, the second condition says that there also the deflection is zero, which is obvious. Now we search the bending at the upper edge of the neck (at the bridge), if the neck has length of 0.4 m counted from the point where it enters the guitar body.

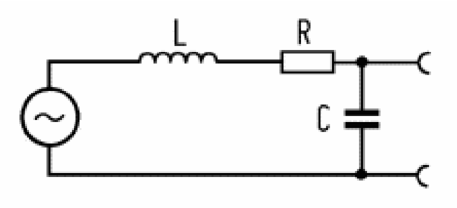
Tip: Calculate the second derivative, integrate and calculate the first integration constant from the first boundary condition. Integrate again and calculate the second integration constant from second boundary condition. Then calculate maximum bending at x=0.4 m.

Exercise 7

Complex numbers / maximum / differential calculus:

A pick-up of an electric guitar is characterized by a transition behavior, which transmits low frequencies directly without amplification or damping. Intermediate frequencies are amplified and high frequencies are damped. The largest amplification occurs at the resonance frequency. Since higher frequencies than the resonance frequency are increasingly damped, the resonance frequency is characteristic for the sound of the guitar. Gibson guitars usually have low resonance frequencies, while Fender guitars have typically high resonance frequencies. This is one of the reasons for the smoother sound of Gibson guitars in contrast to 'harder' sound of Fender guitars. Another reason for the softer Gibson sound is that Gibson uses mostly Humbucker (double coil) type pick-ups, which usually have a lower resonance frequency than single coil pick-ups typical for Fender guitars.

A pick-up can be described by an electrical substitute circuit



containing an ohmic resistance, a capacity and an inductivity.

The ohmic resistance R, the capacity C as well as the inductivity L can be determined from measurements. In the table below results of such measurements are given for a single coil and for a humbucker pick-up (Note: the capacity unit pF $=10^{-12}$F).

	R [Ohm]	L [Henry]	C [pF)
Single Coil	7160	3.8	130
Humbucker	16000	12.6	100

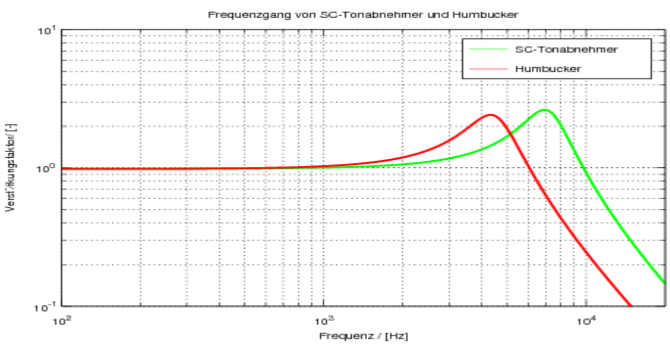
Tab.: Electric data for single coil and humbucker.

For the amplification factor A of the pick-up we find from Kirchhoff's rules depending on angular frequency $\omega = 2\pi f$

$$A = \left|\frac{1}{(R+j\omega L)\cdot j\omega C+1}\right| = \frac{1}{|(R+j\omega L)\cdot j\omega C+1)|}$$

Find the resonance frequency for both pick-ups as characterized in the table above. How can we calculate the resonance frequency approximately, if R and C are small?

Tip: The amplification formula is given with complex numbers. Thus as a first step one has to determine the absolute value of a complex number. To do this order the nominator according to real and complex parts, multiply denominator and nominator with conjugate complex number of the nominator and find the square root, which yields the absolute value. Then differentiate the function with respect to angular frequency using product and chain rules to get the angular frequency at maximum amplitude. Finally calculate frequency f and compare with the pictures given below.



Exercise 8:

Linear equations with two unknowns:

A musician plays electric guitar and also electric bass. He's playing for long time already and owns many guitars. Now he wants to replace the strings of all of his 9 instruments. A guitar string costs 1 Euro, a bass string is more expensive and costs 2 Euros. In the music shop he has to pay 62 Euros. How many guitars and how many bass guitars has the guy in his stock?

Tip: Set number of guitars to unknown variable x and number of bass guitars to unknown y. Then translate the information given in the text above into mathematical language. E.g., number of instruments given is 9 means $x+y=9$. Find an analogous expression for the information related to the price of the strings. Then you will get another equation also involving the unknown x and y. Find an expression for x from the simplest one of the equations and introduce this into the other equation. Thus you will get rid of the x there and end up with an equation only containing y . Now you can solve for y and when you found it you can introduce the value of y into the first equation to find the x as well.

Exercise 9:

Linear equations with three unknowns:

Another musician also playing e-guitar and e-bass owns some guitars, some 4 string bass guitars and also some 5 string bass guitars. In total he owns 9 instruments. He wants to perform an overall maintenance of his instruments by replacing all of the strings as well as all of the pickups. Each guitar string is 1 Euro, each bass string 2 Euros. For the strings he has to pay in total 68 Euros. The pickups are much more expensive. A guitar pickup costs 30 Euros and he needs three of them for each of the guitar. A bass pickup costs 40 Euro and each of the bass guitars carries two of them. In total he has to pay 760 Euros for the pickups. Find out how many guitars and how many 4-string as well as 5-string basses the person owns.

Tip: Use the same procedure as in previous exercise, but introduce a third variable z for the number of 5-string basses. Then also transform the conditions given for string price and pickup price into two additional equations. Find an expression for x from one of these equations and introduce it into the other equations. Thus you will get rid of the x. Then simplify the expressions you got by ordering according to y and z. Now you have reduced the problem to one you already solved in last exercise. Proceed in same way.

Exercise 10:

Linear equations with three unknowns.

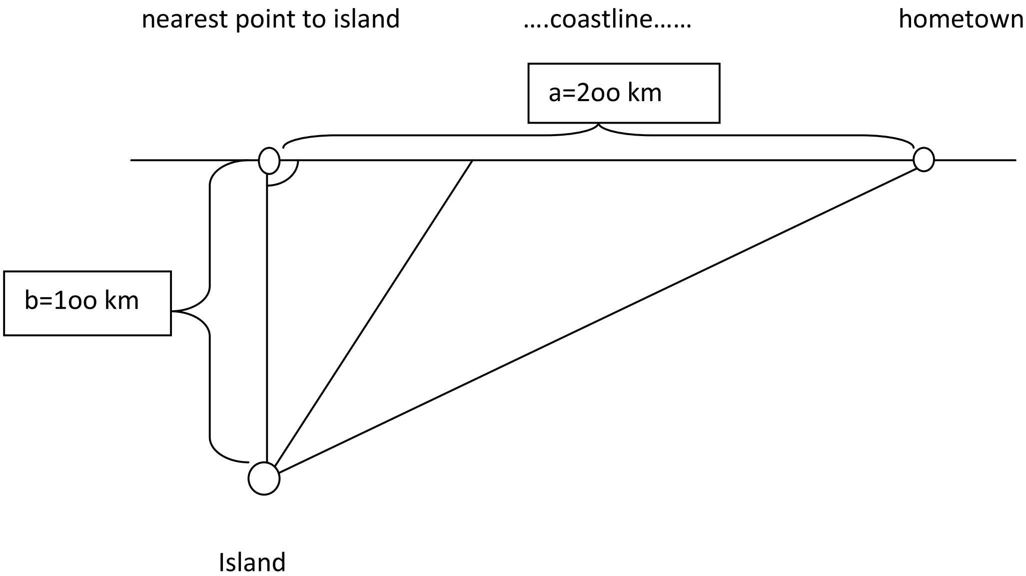
A person wants to make a party. He invites 10 friends. Some of them would like to drink beer, some wine and some of them would like to drink juice. He himself prefers beer. For the beer drinkers he buys 3 liters of beer per person. For the wine drinkers he buys 1 liter of wine per person and for those preferring juice 2 liters of juice for each of them. In total he has to buy 24 liters of soft and alcoholic drinks. He further knows that all beer drinkers and also the wine drinkers smoke. Thus he buys 4 packets of cigarettes assuming that each of them will consume half a packet. How many beer drinkers , wine drinkers and anti-alcoholic people will join the party?

Tip: Similar as exercise 9. Don't forget to add the host of the party to total number of participants.

Exercise 11:

Differential calculus / finding an optimum:

A musician lives in a city at the coast of the sea and wants to give a concert on an island situated as shown in the sketch below . He could travel directly by boat or travel some distance along the coast and then take the boat. The coast line extends more or less as an approximately straight line. Times are uncertain and there is a certain risk R_L per kilometer to get robbed by bandits at land, but, there is an even higher risk R_S to get robbed by pirates on the sea. The distance from his home town to the coast point opposite to the island is 200 km, the distance from this nearest point to the island is 100 km. The risk to become a victim of pirates is about 2.236 (= √5) times higher than at land along the coast. At which point the musician should take the boat to minimize his total risk to get robbed.



Tip: Introduce variable x with some value between 0 and 1 as a fraction of distance a and write x·a for the distance of nearest point to embarkation point. Then as a consequence the distance between embarkation point and hometown is a·(1-x). Then set up a function y(x) for the total risk which is risk per kilometer times travelling path on land and on sea. Don't forget to introduce the enhanced risk for travel by boat. This function should be minimized. Find the value of x for the minimum of the risk function. This exercise shows that some mathematical knowledge can help to extend a musicians life!

Exercise 12:

Some things to prove:

An annotation in advance on proving. Besides its usefulness a big benefit of mathematics is that it cannot be abused for ethnic or racist discrimination. Also male and female human beings are treated in the same fair way in mathematics. No teacher can introduce his personal mindset and personal opinion or prejudice against some kind of people into the tasks and into the evaluations of the results of a test or exam. The reason is that mathematical facts can be proven objectively and everybody can check the results whether they are right or wrong. Thus mathematics has an inherent democratic nature. Therefore proving mathematical facts is very important. If you know how to prove correctly everybody has to accept your results. Additionally you have found some fix point in the world, which will also stabilize you personally, since you share this knowledge with all the other people dealing with maths. Further regarding the fact you have proven, you are at the same level with your teacher and with all mathematically interested persons all over the world. This provides some equality between men. This is something needed very urgently in nowadays society.

A useful method for proving is the technique of mathematical induction. This is often used for some relations regarding natural numbers. The idea behind is to test a mathematical thesis by one or two examples and then show in general terms that the fact holds for the next number assuming it is valid for the previous one.

An example: The sum of the first n natural numbers is $n(n+1)/2$.

First step: Show validity for one or two examples: For $n=1$ the sum according to the formula is $1(1+1)/2 = 1$ which is true. Similar for $n=2$ the sum is $1+2= 2(2+1)/2=3$, which is also true. For $n=3$ we have $1+2+3=6= 3(3+1)/2=6$ and so on ….
Second step: Now we write the formula for n which is our assumption

$1+2+3+\ldots+ n = n(n+1)/2$ About this assumption we know that it is true at least for n=1 and 2 and 3.

Third step: Then we write the formula for n+1 and show that it holds based on the assumption in step 2.

$1+2+3+4+\ldots+n+(n+1) = (n+1)(n+2)/2$

To do this we can introduce the assumption from step 2 into the last equation by replacing $1+2+3+\ldots+n$ on left hand side by the proposition given in step 2. I.e.,

$n(n+1)/2 \quad + \quad (n+1)=(n+1)(n+2)/2$

On left hand side we can factor out the expression $(n+1)$ and get

$(n+1)(n/2+1) = (n+1)(n/2+2/2) =(n+1)(n+2)/2$

Thus we have got an identity which is always a true fact and thus the formula holds also for (n+1), if it holds for n. Since we have found by our first test that things are ok for n=1 and n=2 and n=3 this also holds for n=4 and so on for any next n.

Here are exercises for own testing:

a) $1 + 3 + 5 + 7 + \ldots + (2n-1) = n^2$

b) $1^2 + 2^2 + 3^2 + 4^2+ \ldots+n^2 = n(n+1)(2n+1)/6$

c) $1^3 + 2^3 + 3^3 + 4^3 + \ldots+n^3 = [1+2+3+4+\ldots+n]^2 = [n(n+1)/2]^2$

d) All number of shape $5^n + 2^{n+1}$ are multiples of 3

e) All numbers of shape $2^{2^n} + 1$ have 7 as last digit, if $n \geq 2$

Tip: Don't be afraid! Try it! You might need some basic knowledge about square and cubic powers of sums and also the rules for calculating with powers, especially in d) and e). Repeat these things from some formula collection! You will definitely need them also in next summer 's exam.

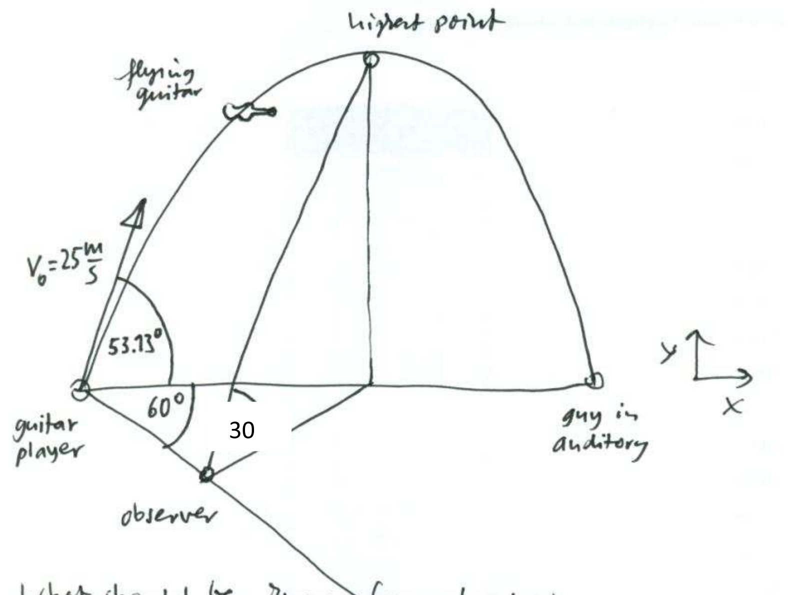
Exercise 13:

Differential calculus and trigonometric functions / also some physics knowledge needed:

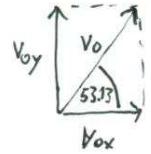
At the end of a rock-concert the guitar player throws his guitar into the auditory. He does this with an initial speed of 25 m/s under an angle of 53.13 degree against horizontal plane. In the auditory a guy catches the guitar exactly at the position where it falls again back to the ground. Another guy in the auditory observes this perfect artistic action. He sees that the highest point of the parabolic path of the flying guitar is passed under an angle of 30 degrees observed from his position. About his position unfortunately we do not know all the data. It is just known that he stands somewhere on a straight line which is inclined by 60 degree against the projection of the parabola of flight onto the ground. To simplify things all of the positions of start point of flight, end point of flight and eye of the observer are in the same plane. Take constant of gravity g as 10 m/s^2.

Please try to find out, how big is the horizontal distance between the guitar player throwing the guitar and the guy in the auditory, who catching the guitar. Find also the distance between player and observer.

Tips: Find the parabola from physical laws as shown below. Calculate the coefficients. Find highest point from differentiation (maximum of parabolic function). Then find distance between projection point of maximum to observer from suitable rectangular triangle. Finally find also the distance from player to observer from law of sines and law of cosines.



What should be known from physics:



$$y = V_{oy} \cdot t - \frac{g}{2} t^2$$

$$x = V_{ox} \cdot t \quad \rightarrow t = \frac{x}{V_{ox}}$$

$$\rightarrow y = \frac{V_{oy}}{V_{ox}} \cdot x - \frac{g}{2 V_{ox}^2} \cdot x^2$$